Intro to Passive Dynamic Walker

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Abstract

This paper is project paper from Modeling & Simulation course of Yonsei Univ. It is the intro to how to simulate passive dynamic walker. In the beginning, it start compass gait and added the knee. Finally, we can simulate the "Kneeded Walker".

1 Introduction

Passive dynamic walkers are simple yet effective models for studying bipedal locomotion. This paper explores the dynamics and simulation of a passive dynamic walker, beginning with the compass gait and extending to the kneaded walker with knees. The addition of knees introduces new degrees of freedom, dynamic coupling, and phases, making the model more complex and closer to human-like walking behavior.

2 Compass Gait

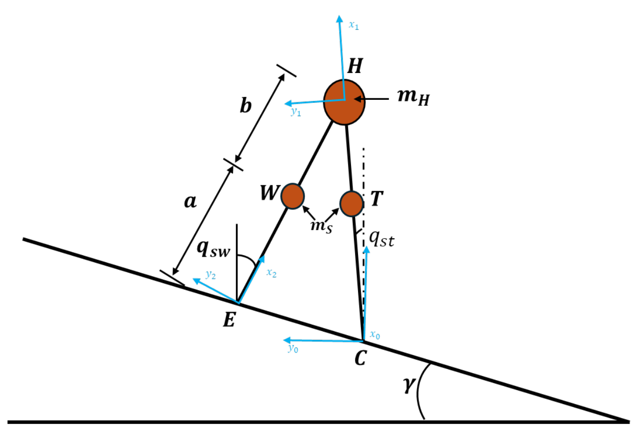

Figure 1: Compass Gait

First, we will define the notation before proceeding. The following five points are defined:

- **C**: Point of Contact
- **T**: Stand leg
- **W**: Swing leg
- **E**: End foot
- **H**: Hip

2.1 Swing Phase

2.1.1 Kinematics

Compass gait consists of two phases. The first phase is the swing phase, which can be analyzed as a pendulum motion. In this phase, the swing leg moves under the influence of gravity while the stance leg remains stationary. This motion resembles the dynamics of an inverted pendulum, making it suitable for simplified analysis. Let the initial x_0, y_0 be aligned with the line moved by the angle q_{st} from the stand leg, keeping x_0 unchanged. Representing the coordinates of all relative points in the 0th frame, they are as follows:

$$
\begin{aligned}
P^0_{CT} &= R^0_1 P^1_{CT} = R^1_0 \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, & P^0_{CH} &= R^0_1 P^1_{CH} = R^1_0 \begin{bmatrix} a+b \\ 0 \\ 0 \end{bmatrix}, \\
P^0_{EH} &= R^0_2 P^2_{EH} = R^2_0 \begin{bmatrix} a+b \\ 0 \\ 0 \end{bmatrix}, & P^0_{WH} &= R^0_2 P^2_{WH} = R^2_0 \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}, \\
P^0_{HE} &= -P^0_{EH}, & P^0_{HW} &= -P^0_{WH}.
\end{aligned}
$$

$$
\text{where } R^1_0 = \begin{bmatrix} \cos(\theta_{st}) & -\sin(\theta_{st}) & 0 \\ \sin(\theta_{st}) & \cos(\theta_{st}) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R^2_0 = \begin{bmatrix} \cos(\theta_{sw}) & -\sin(\theta_{sw}) & 0 \\ \sin(\theta_{sw}) & \cos(\theta_{sw}) & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

To derive the equation of motion (EOM) for the swing leg, the following four vectors can be expressed as follows:

$$
\begin{aligned}
P^0_{CT} &= R^0_1 P^1_{CT} = R^1_0 \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, & P^0_{CH} &= R^0_1 P^1_{H} = R^1_0 \begin{bmatrix} a+b \\ 0 \\ 0 \end{bmatrix}, \\
P^0_{CW} &= P^0_{CH} + P^0_{HE}, & P^0_{CW} &= P^0_{CH} + P^0_{HW}.
\end{aligned}
$$

Now, from this position, we can calculate the velocity and acceleration as follows:

$$
\begin{aligned}
V_T &= V_C + \omega^{ST} \times P^0_{CT}, & V_H &= V_C + \omega^{ST} \times P^0_{CH}, \\
V_W &= V_H + \omega^{SW} \times P^0_{HW}, & V_E &= V_H + \omega^{SW} \times P^0_{HE},
\end{aligned}
$$

$$
\begin{aligned}
a_T &= a_C + \alpha^{ST} \times P^0_{CT} + \omega^{ST} \times (\omega^{ST} \times P^0_{CT}), & a_H &= a_C + \alpha^{ST} \times P^0_{CH} + \omega^{ST} \times (\omega^{ST} \times P^0_{CH}), \\
a_W &= a_H + \alpha^{SW} \times P^0_{HW} + \omega^{SW} \times (\omega^{SW} \times P^0_{HW}), & a_E &= a_H + \alpha^{SW} \times P^0_{HE} + \omega^{SW} \times (\omega^{SW} \times P^0_{HE}).
\end{aligned}
$$

2.1.2 Euler-Lagrangian Equation

The Euler-Lagrangian equation derives the equations of motion (EOM) by defining the Lagrangian L as the difference between the kinetic energy (K) and potential energy (P). The EOM can then be expressed as:

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau,
$$

where $L = K - P$, K denotes the kinetic energy, and P represents the potential energy. Here, τ represents external forces or torques acting on the system.

The kinetic energy is given by:

$$
K = \frac{1}{2} m_s V_T^T V_T + \frac{1}{2} m_H V_H^T V_H + \frac{1}{2} m_s V_W^T V_W + \frac{1}{2} \omega_{ST}^T \mathcal{I} \omega_{ST} + \frac{1}{2} \omega_{SW}^T \mathcal{I} \omega_{SW},
$$

The potential energy is given by:

$$
P = m_s g P^0_{CT,x} + m_H g P^0_{CH,x} + m_s g P^0_{CW,x},
$$

Let $q = [q_{ST}, q_{SW}]$. Thus, the Euler-Lagrangian (E-L) equations can be written as:

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_{ST}}}\right) - \frac{\partial L}{\partial q_{ST}},
$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_{SW}}}\right) - \frac{\partial L}{\partial q_{SW}}.$$

The equations can be represented in terms of M, C, and G as follows:

$$M = \begin{bmatrix} \mathcal{I}_{ST} + (a+b)^2 m_H + (2a^2 + 2ab + b^2) m_S & -b(a+b) m_S \cos(q_{ST} - q_{SW}) \\ -b(a+b) m_S \cos(q_{ST} - q_{SW}) & \mathcal{I}_{SW} + b^2 m_S \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & -b(a+b) m_S \sin(q_{ST} - q_{SW}) \dot{q}_{SW} \\ b(a+b) m_S \sin(q_{ST} - q_{SW}) \dot{q}_{ST} & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} -g\big(b(m_H + m_S) + a(m_H + 2m_S)\big) \sin(q_{ST}) \\ bgm_S \sin(q_{SW}) \end{bmatrix}.$$

Now, we have

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau$$

2.2 Heel Strike

The second phase is the "Heel Strike". In this phase, the collision occurs in a closed system where momentum is conserved. If momentum is conserved about one axis, it is also conserved about any other axis. Before **contact**, we had N states. Now, we extend to $N+2$ states.

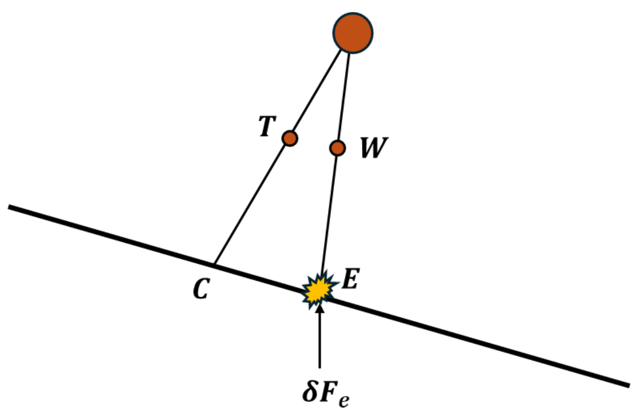

Figure 2: Contact

The extended coordinates are:

$$P_c = \begin{bmatrix} P_{C,x} \\ P_{C,y} \\ 0 \end{bmatrix}, \quad \dot{P}_c = \begin{bmatrix} \dot{P}_{C,x} \\ \dot{P}_{C,y} \\ 0 \end{bmatrix}.$$

Then, our q now also has $N+2$ elements:

$$q_e = [q_{ST}, q_{SW}, P_{C,x}, P_{C,y}].$$

2.2.1 Dynamic Equation

At the moment of contact, forces are applied due to the collision. By the impulse-momentum theorem, the momentum just before (t^-) and just after (t^+) the contact must be conserved. To analyze this interaction, we introduce two additional states at the point of contact.

The extended state vector is defined as:

$$q_e = [q_{ST}, q_{SW}, P_{C,x}, P_{C,y}].$$

The dynamic equation of the system becomes:

$$M_e(q_e)\ddot{q_e} + C_e(q_e, \dot{q_e})\dot{q_e} + G(q_e) = B\tau + J^T \cdot \delta F_e,$$

where δF_e is converted to $\delta\tau$ by the Jacobian (J).

Thus, the control input becomes:

$$B\tau + \delta\tau.$$

For an instantaneous moment, we integrate the equation:

$$\int M_e(q_e)\ddot{q_e}dt + \int C_e(q_e, \dot{q_e})\dot{q_e}dt + \int G(q_e)dt = \int B\tau dt + \int \delta\tau dt.$$

Each term is evaluated as follows:

1. $\ddot{q_e}$:

$$\int \ddot{q_e}dt = \int_-^+ \frac{d\dot{q_e}}{dt}dt = \int_-^+ d\dot{q_e} \rightarrow \dot{q_e}^+ - \dot{q_e}^-.$$

2. $\dot{q_e}$:

$$\int \dot{q_e}dt = \int_-^+ \frac{dq_e}{dt}dt = \int_-^+ dq_e \rightarrow q_e^+ - q_e^- = 0,$$

as there is no change in angles during impact.

3. $G(q_e)$:

$$\int_-^+ G(q_e)dt = 0,$$

assuming that there are no gravitational effects during the impact.

4. Actuator torque ($B\tau$):

$$\int_-^+ B\tau dt = 0,$$

assuming no actuator torque during impact.

5. Impact force ($\delta\tau$):

$$\int_-^+ \delta\tau dt = \tau = J^T F_e.$$

Hence, the final equation becomes:

$$M_e(q_e)\dot{q_e}^+ - M_e(q_e)\dot{q_e}^- = J^T F_e.$$

We assume an additional condition:

$$V_e = J\dot{q_e}^+ = 0,$$

, which represents the non-slip condition.

Finally, the matrix equation can be expressed as:

$$\begin{bmatrix} M_e(q_e) & -J^T \\ J & 0 \end{bmatrix} \begin{bmatrix} \dot{q_e}^+ \\ F_e \end{bmatrix} = \begin{bmatrix} M_e(q_e)\dot{q_e}^- \\ 0 \end{bmatrix}. \tag{1}$$

where:

$$\dot{q_e}^- = \begin{bmatrix} \dot{q_{ST}}^- \\ \dot{q_{SW}}^- \\ 0 \\ 0 \end{bmatrix}, \quad F = \mu F_N.$$

Using Equation (1), we can solve for $\dot{q_e}^+$ and F_e. These values can be computed using a simple matrix inversion, as shown below:

$$\begin{bmatrix} \dot{q_e}^+ \\ F_e \end{bmatrix} = \begin{bmatrix} M_e(q_e) & -J^T \\ J & 0 \end{bmatrix}^{-1} \begin{bmatrix} M_e(q_e)\dot{q_e}^- \\ 0 \end{bmatrix}.$$

2.3 Simulation

The simulation of the compass gait is shown in Figure 3. The motion of the compass gait is analyzed based on its trajectory, angular velocity, and phase portraits of θ_{ST} and θ_{SW}.

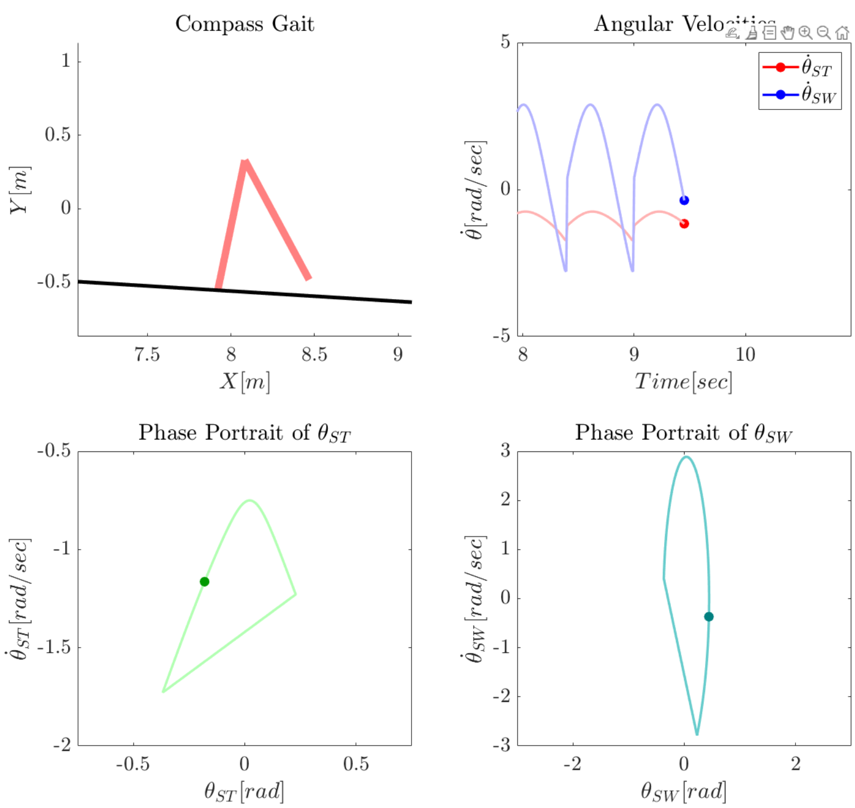

Figure 3: Simulation results of the compass gait

From the simulation results:

- The trajectory in the X-Y plane demonstrates the periodic motion of the compass gait. The swing leg and stance leg alternate to maintain balance and propulsion.
- The angular velocity plot shows the periodic nature of $\dot{\theta}_{ST}$ and $\dot{\theta}_{SW}$. Peaks correspond to the dynamic exchange between the stance and swing phases.
- The phase portraits illustrate the stability and periodic behavior of the system. Both θ_{ST} and θ_{SW} exhibit closed-loop trajectories, which indicate stable limit cycles.

3 Kneeded Walker Model

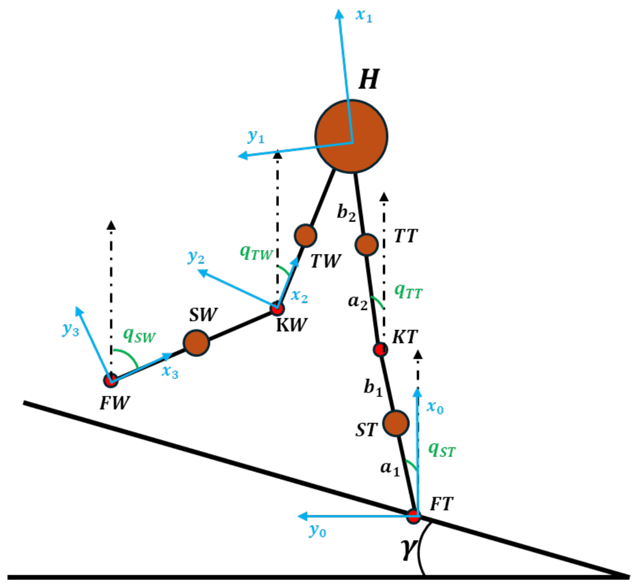

Figure 4: Kneeded Walker

Now, let us consider the dynamic walker with knees. With the addition of knees, the two links of the compass gait model are extended to four links, and an additional phase is introduced. This modification significantly increases the complexity of the model, as the interaction between links must now account for the bending and extension of the knees.

Following the same approach for defining notations, we define the components of the kneaded walker as:

- **T**: Thigh
- **K**: Knee
- **S**: Shank
- **F**: Foot

(Note: The second letter is **T** for the stand leg and **W** for the swing leg.)

3.1 Kinematics

We define every vector relative to the reference frame $x_0 - y_0$. Then, the actual point positions can be represented as the summation of each vector.

For Stand Foot:

$$P_{FT\to ST} = R_1^0 \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}, \quad P_{FT\to KT} = R_1^0 \begin{bmatrix} a_1 + b_1 \\ 0 \\ 0 \end{bmatrix},$$

$$P_{FT\to TT} = R_1^0 \begin{bmatrix} a_1 + b_1 + a_2 \\ 0 \\ 0 \end{bmatrix}, \quad P_{FT\to H} = R_1^0 \begin{bmatrix} a_1 + b_1 + a_2 + b_2 \\ 0 \\ 0 \end{bmatrix}.$$

For Swing Foot:

$$P_{TW\to H} = R_2^0 \begin{bmatrix} b_2 \\ 0 \\ 0 \end{bmatrix}, \quad P_{KW\to H} = R_2^0 \begin{bmatrix} a_2 + b_2 \\ 0 \\ 0 \end{bmatrix},$$

$$P_{SW\to KW} = R_3^0 \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix}, \quad P_{FW\to KW} = R_3^0 \begin{bmatrix} a_1 + b_1 \\ 0 \\ 0 \end{bmatrix}.$$

Rotation Matrices:

$$R_1^0 = \begin{bmatrix} \cos(q_{ST}) & -\sin(q_{ST}) & 0 \\ \sin(q_{ST}) & \cos(q_{ST}) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_2^0 = \begin{bmatrix} \cos(q_{TW}) & -\sin(q_{TW}) & 0 \\ \sin(q_{TW}) & \cos(q_{TW}) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_3^0 = \begin{bmatrix} \cos(q_{SW}) & -\sin(q_{SW}) & 0 \\ \sin(q_{SW}) & \cos(q_{SW}) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have 4 variables to describe the system:

$$q_{ST},\ q_{KT},\ q_{TW},\ q_{SW}.$$

The linear velocity at various points is given by the following equations:

$$\begin{aligned} V_{FT} &= 0 \quad \textit{(no slip condition)} & V_{KW} &= V_H + \omega_{TW} \times P_{H\to KW} \\ V_{ST} &= V_{FT} + \omega_{ST} \times P_{FT\to ST} & V_{SW} &= V_{KW} + \omega_{SW} \times P_{KW\to SW} \\ V_{KT} &= V_{FT} + \omega_{ST} \times P_{FT\to KT} & V_{FW} &= V_{KW} + \omega_{SW} \times P_{KW\to FW} \\ V_{TT} &= V_{KT} + \omega_{TT} \times P_{KT\to TT} & V_{TW} &= V_H + \omega_{TW} \times P_{H\to TW} \\ V_H &= V_{KT} + \omega_{TT} \times P_{KT\to H} \end{aligned}$$

and the linear acceleration at various points is given by the following equations:

$$\begin{aligned} a_{FT} &= V_{FT} = 0 \\ a_{ST} &= a_{FT} + \alpha_{ST} \times P_{FT\to ST} + \omega_{ST} \times (\omega_{ST} \times P_{FT\to ST}) \\ a_{KT} &= a_{FT} + \alpha_{ST} \times P_{FT\to KT} + \omega_{ST} \times (\omega_{ST} \times P_{FT\to KT}) \\ a_{TT} &= a_{KT} + \alpha_{TT} \times P_{KT\to TT} + \omega_{TT} \times (\omega_{TT} \times P_{KT\to TT}) \\ a_H &= a_{KT} + \alpha_{TT} \times P_{KT\to H} + \omega_{TT} \times (\omega_{TT} \times P_{KT\to H}) \end{aligned}$$

$$\begin{aligned} a_{TW} &= a_H + \alpha_{TW} \times P_{H\to TW} + \omega_{TW} \times (\omega_{TW} \times P_{H\to TW}) \\ a_{KW} &= a_H + \alpha_{TW} \times P_{H\to KW} + \omega_{TW} \times (\omega_{TW} \times P_{H\to KW}) \\ a_{SW} &= a_{KW} + \alpha_{SW} \times P_{KW\to SW} + \omega_{SW} \times (\omega_{SW} \times P_{KW\to SW}) \\ a_{FW} &= a_{KW} + \alpha_{SW} \times P_{KW\to FW} + \omega_{SW} \times (\omega_{SW} \times P_{KW\to FW}) \end{aligned}$$

3.2 Euler-Lagrangian equation

Now, we can calculate the Euler-Lagrangian equation. First, the kinetic energy is expressed as:

$$\begin{aligned} K &= \frac{1}{2}\mathcal{I}\omega^2 \\ &= \frac{1}{2}\left(\mathcal{I}_c + ml^2\right)\omega^2 \\ &= \frac{1}{2}\mathcal{I}_c\omega^2 + \frac{1}{2}m(l\omega)^2 =: \frac{1}{2}\mathcal{I}_c\omega^2 + \frac{1}{2}mv^2 \end{aligned}$$

Assuming the symmetry conditions $m_{ST} = m_{SW} = m_S$ and $m_{TT} = m_{TW} = m_T$, the total kinetic energy is given by:

$$\begin{aligned} K = &\frac{1}{2}m_S V_{ST}^T V_{ST} + \frac{1}{2}m_T V_{TT}^T V_{TT} + \frac{1}{2}m_H V_H^T V_H + \frac{1}{2}m_T V_{TW}^T V_{TW} + \frac{1}{2}m_S V_{SW}^T V_{SW} \\ &+ \frac{1}{2}\omega_{ST}^T \mathcal{I}_K \omega_{ST} + \text{(Swing Leg Part)}. \end{aligned} \tag{2}$$

$\mathcal{I}_K$ is the modified inertia tensor derived from $\mathcal{I}_S$ and $\mathcal{I}_T$.

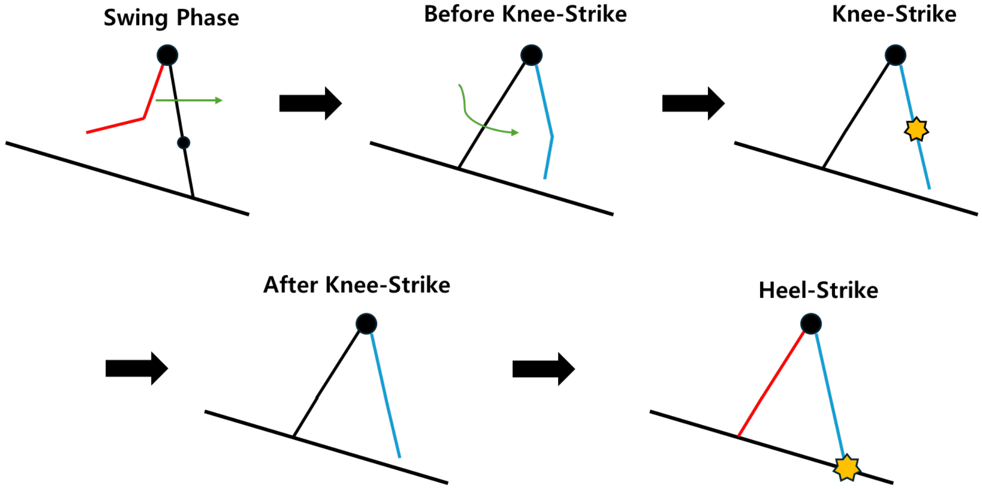

Figure 5: Passive Walker Phase

As shown in Figure 5, the passive dynamic walker moves in two distinct phases. Before the knee-strike, it swings like a 2-link pendulum. After the knee-strike occurs, it transitions to a compass-gait motion.

Therefore, we can divide the Swing Leg Part in Equation (2) into two cases: **before knee-strike** and **after knee-strike**.

Before Knee Strike:

$$\frac{1}{2}\omega_{TW}^T \mathcal{I}_T \omega_{TW} + \frac{1}{2}\omega_{SW}^T \mathcal{I}_S \omega_{SW}$$

After Knee Strike:

$$\frac{1}{2}\omega_{SW}^T \mathcal{I}_K \omega_{SW}$$

3.2.1 Euler-Lagrangian equation of Before Knee Strike

We aim to derive and calculate the Euler-Lagrangian equations for each phase of the passive dynamic walker.

First, let us derive the Euler-Lagrangian equation for the phase **before the knee strike**.

The Kinetic energy K is:

$$\begin{aligned} K = &\frac{1}{2}m_S V_{ST}^T V_{ST} + \frac{1}{2}m_T V_{TT}^T V_{TT} + \frac{1}{2}m_H V_H^T V_H + \frac{1}{2}m_T V_{TW}^T V_{TW} + \frac{1}{2}m_S V_{SW}^T V_{SW} \\ &+ \frac{1}{2}\omega_{ST}^T \mathcal{I}_K \omega_{ST} + \frac{1}{2}\omega_{TW}^T \mathcal{I}_T \omega_{TW} + \frac{1}{2}\omega_{SW}^T \mathcal{I}_S \omega_{SW} \end{aligned}$$

and the potential energy P is:

$$P = m_S g P_{FT\to ST,x} + m_T g P_{FT\to TT,x} + m_H g P_{FT\to H,x} + m_T g P_{FT\to TW,x} + m_S g P_{FT\to SW,x}$$

As in the case of the compass gait, we calculate the Euler-Lagrangian (E-L) equation and find the M, C, and G matrices. Using these matrices, we can solve for $\ddot{q}$, which represents the system's acceleration.

The dynamics of the system are represented as:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau,$$

By rearranging the equation, we can compute $\ddot{q}$ as:

$$\ddot{q} = M(q)^{-1}\big(\tau - C(q,\dot{q})\dot{q} - G(q)\big).$$

3.2.2 Euler-Lagrangian equation after Knee-Strike

After the knee strike, the system transitions to compass gait dynamics. The kinetic energy K is now given by:

$$\begin{aligned}K = &\frac{1}{2}m_S V_{ST}^T V_{ST} + \frac{1}{2}m_T V_{TT}^T V_{TT} + \frac{1}{2}m_H V_H^T V_H \\ &+ \frac{1}{2}m_T V_{TW}^T V_{TW} + \frac{1}{2}m_S V_{SW}^T V_{SW} \\ &+ \frac{1}{2}\omega_{ST}^T \mathcal{I}_K \omega_{ST} + \frac{1}{2}\omega_{SW}^T \mathcal{I}_K \omega_{SW}.\end{aligned}$$

The potential energy P same as in the phase before the knee strike:

$$P = m_S g P_{FT\to ST,x} + m_T g P_{FT\to TT,x} + m_H g P_{FT\to H,x} + m_T g P_{FT\to TW,x} + m_S g P_{FT\to SW,x}.$$

The Euler-Lagrangian equation is:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau,$$

where $M(q)$, $C(q,\dot{q})$, and $G(q)$ are recalculated for the new dynamics.

3.3 Knee Strike

For convenience, we now describe the 3-variable version (q_{ST}, q_{TW}, q_{SW}).

In the compass gait, we observed that the heel strike involves an external force at the stand foot, conserving the angular momentum of the swing leg. However, in the case of knee strike, it is more appropriate to consider it as an external impulse. Therefore, the angular momentum is not conserved about the knee.

To address this, we apply the impulse-momentum theory, which states that the momentum before and after the collision remains equal. This can be expressed as (at FT and H; for convenience, all subscripted references are with respect to the origin of the position):

$$\begin{aligned}H_{FT}^- &= m_S(\vec{P_{ST}} \times v_{ST}^-) + m_T(\vec{P_{TT}} \times v_{TT}^-) + m_H(\vec{P_H} \times v_H^-) \\ &\quad + m_T(\vec{P_{TW}} \times v_{TW}^-) + m_S(\vec{P_{SW}} \times v_{SW}^-) \\ H_{FT}^+ &= m_S(\vec{P_{ST}} \times v_{ST}^+) + m_T(\vec{P_{TT}} \times v_{TT}^+) + m_H(\vec{P_H} \times v_H^+) \\ &\quad + m_T(\vec{P_{TW}} \times v_{TW}^+) + m_S(\vec{P_{SW}} \times v_{SW}^+) \\ H_H^- &= m_T(\vec{P_{TW}} \times v_{TW}^-) + m_S(\vec{P_{SW}} \times v_{SW}^-) \\ H_H^+ &= m_T(\vec{P_{TW}} \times v_{TW}^+) + m_S(\vec{P_{SW}} \times v_{SW}^+).\end{aligned}$$

For a more advanced version, such as modeling in 3D space, we can extend this to include angular momentum contributions:

$$\begin{aligned}\bar{H}_{FT}^- &= H_{FT}^- + (\mathcal{I}_K \cdot \omega_{ST}^-) + (\mathcal{I}_T \cdot \omega_{TW}^-) + (\mathcal{I}_S \cdot \omega_{SW}^-) \\ \bar{H}_{FT}^+ &= H_{FT}^+ + (\mathcal{I}_K \cdot \omega_{ST}^+) + (\mathcal{I}_T \cdot \omega_{TW}^+) + (\mathcal{I}_S \cdot \omega_{SW}^+) \\ \bar{H}_H^- &= H_H^- + (\mathcal{I}_T \cdot \omega_{TW}^-) + (\mathcal{I}_S \cdot \omega_{SW}^-) \\ \bar{H}_H^+ &= H_H^+ + (\mathcal{I}_T \cdot \omega_{TW}^+) + (\mathcal{I}_S \cdot \omega_{SW}^+).\end{aligned}$$

- At FT, $\mathcal{I}_K$ is the moment of inertia at the knee. It accounts for the swing leg before knee-strike and the combined leg after the knee locks.

- When the leg straightens (knee-locks), the new center of mass (P_{COM}) for the straightened leg is calculated as:

$$\left(\sum m_i\right) P_{COM} = \sum_i \left(m_i \cdot P_{COM,i}\right).$$

Using P_{COM}, we calculate the updated $\mathcal{I}_K$.

To establish a relationship between $\begin{bmatrix}\omega_{ST}^- \\ \omega_{SW}^-\end{bmatrix}$ and $\begin{bmatrix}\omega_{ST}^+ \\ \omega_{SW}^+\end{bmatrix}$, we define the Jacobian:

$$\begin{bmatrix}\frac{\partial H_{FT}^+}{\partial \omega_{ST}^+} & \frac{\partial H_{FT}^+}{\partial \omega_{SW}^+} \\ \frac{\partial H_H^+}{\partial \omega_{ST}^+} & \frac{\partial H_H^+}{\partial \omega_{SW}^+}\end{bmatrix} \begin{bmatrix}\omega_{ST}^+ \\ \omega_{SW}^+\end{bmatrix} = \begin{bmatrix}H_{FT}^- \\ H_H^-\end{bmatrix}.$$

Let:

$$A = \begin{bmatrix} \frac{\partial H_{FT}^{+}}{\partial \omega_{ST}^{+}} & \frac{\partial H_{FT}^{+}}{\partial \omega_{SW}^{+}} \\ \frac{\partial H_{H}^{+}}{\partial \omega_{ST}^{+}} & \frac{\partial H_{H}^{+}}{\partial \omega_{SW}^{+}} \end{bmatrix}, \quad B = \begin{bmatrix} H_{FT}^{-} \\ H_{H}^{-} \end{bmatrix}.$$

Here, B contains ω_{ST}^{-}, ω_{TW}^{-}, and ω_{SW}^{-}. Solving this system allows us to find ω_{ST}^{+} and ω_{SW}^{+}. In other words, the Jacobian represents a mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$.

$$A \begin{bmatrix} \omega_{ST}^{+} \\ \omega_{SW}^{+} \end{bmatrix} = B\left(\omega_{ST}^{-}, \omega_{TW}^{-}, \omega_{SW}^{-}\right)$$

Hence, we can calculate $\left[\omega_{ST}^{+}, \omega_{SW}^{+}\right]^T$ from Matlab with A and B.

3.4 Heel Strike

The heel strike follows the same principles as the compass gait. Since the only external force occurs at the point of impact, no external moments are created around this point, and external torques act on the system. The state variables are reduced to:

$$X = \left[q_{ST}, q_{SW}, \dot{q}_{ST}, \dot{q}_{SW}\right]^T.$$

At the heel strike contact, we must map $\dot{q}^{-} \rightarrow \dot{q}^{+}$.

For the compass-gait model, the following matrix relationship was derived:

$$\begin{bmatrix} M_e(q_e) & -J^T \\ J & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_e^{+} \\ F_e \end{bmatrix} = \begin{bmatrix} M_e(q_e)\dot{q}_e^{-} \\ 0 \end{bmatrix}.$$

Here:

$$\dot{q}_e^{-} = \left[\dot{q}_{ST}^{-}, \dot{q}_{TW}^{-}, \dot{q}_{SW}^{-}, 0, 0\right]^T.$$

3.5 Swap Foot (Leg) after Obtaining $\dot{q}_e^{+} \in \mathbb{R}^5$

Swapping the foot (or leg) after obtaining $\dot{q}_e^{+}$ is straightforward: simply swap $q_{SW} \rightarrow q_{ST}$.

3.6 Simulation

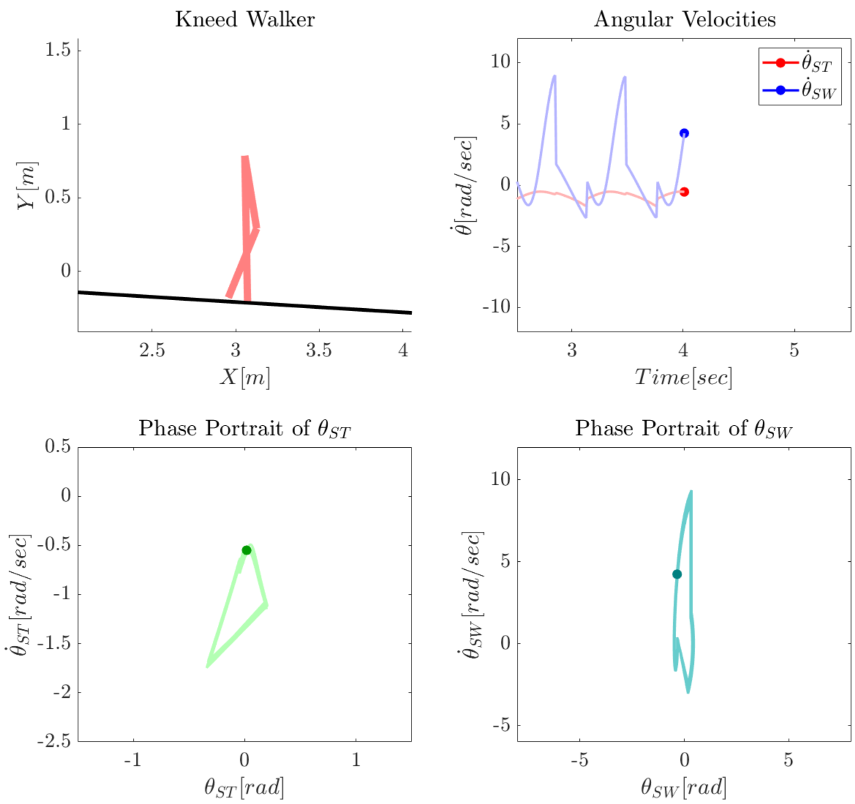

Figure 6: Simulation results of Kneeded Walker

The simulation results of the Kneeded Walker, as shown in Figure 6, demonstrate the following key observations:

1. **Trajectory** ($X - Y$ **Plane**):
   - The walker exhibits periodic and human-like motion in the $X - Y$ plane.
   - The swing leg (red line) follows a curved trajectory, while the stance leg (black line) maintains stability and supports forward movement.
2. **Angular Velocities** ($\dot{\theta}_{ST}$ **and** $\dot{\theta}_{SW}$):
   - The stance leg angular velocity, $\dot{\theta}_{ST}$, shows smoother dynamics with smaller oscillations, reflecting its stabilizing role.
   - The swing leg angular velocity, $\dot{\theta}_{SW}$, displays larger oscillations, representing its active role in forward propulsion.
3. **Phase Portrait of** θ_{ST}:
   - The closed-loop trajectory indicates stable and periodic motion for the stance leg.
   - This phase portrait highlights the energy exchange during the stance phase.
4. **Phase Portrait of** θ_{SW}:
   - The swing leg's phase portrait reveals a larger and more complex loop, reflecting its dynamic role in maintaining forward motion.
   - The closed-loop structure ensures stability in the walking gait.

In summary, the Kneeded Walker simulation confirms stable, periodic, and human-like walking dynamics, with a clear interplay between the stance and swing legs to achieve forward movement and balance.